

数列中的放缩专题参考答案

1.

解：(1) $4S_n = (2n-1)a_{n+1} + 1$

$$\therefore 4S_{n-1} = (2n-3)a_n + 1 \quad (n \geq 2)$$

$$\therefore 4a_n = (2n-1)a_{n+1} - (2n-3)a_n \quad (n \geq 2)$$

$$\text{即 } (2n+1)a_n = (2n-1)a_{n+1} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{2n+1}{2n-1}$$

$$\therefore \frac{a_n}{a_{n-1}} = \frac{2n-1}{2n-3}, \frac{a_{n-1}}{a_{n-2}} = \frac{2n-3}{2n-5}, \dots, \frac{a_3}{a_2} = \frac{5}{3}$$

$$\therefore \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \dots \cdot \frac{a_3}{a_2} = \frac{2n-1}{2n-3} \cdot \frac{2n-3}{2n-5} \cdot \dots \cdot \frac{5}{3} \text{ 即 } \frac{a_n}{a_2} = \frac{2n-1}{3} \quad (n \geq 2)$$

$$\therefore a_n = \frac{2n-1}{3}a_2, \text{ 由 } 4S_n = (2n-1)a_{n+1} + 1 \text{ 令 } n=1 \text{ 可得:}$$

$$4S_1 = a_2 + 1 \Rightarrow a_2 = 3$$

$$\therefore a_n = 2n-1 \quad (n \geq 2), \text{ 验证 } a_1 = 1 \text{ 符合上式}$$

$$\therefore a_n = 2n-1 \quad S_n = n^2$$

$$(2) \text{ 由 (1) 得: } b_n = \frac{1}{(2n-1)\sqrt{n^2}} = \frac{1}{n(2n-1)} \quad b_1 = 1$$

$$\text{可知当 } n \geq 2 \text{ 时, } b_n = \frac{1}{n(2n-1)} < \frac{1}{n(2n-2)} = \frac{1}{2n(n-1)} = \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$\therefore T_n = b_1 + b_2 + \dots + b_n < b_1 + \frac{1}{2} \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \right]$$

$$= 1 + \frac{1}{2} \left(1 - \frac{1}{n} \right) < \frac{3}{2}$$

不等式得证

2.解: (1) $a_{n+1} = 2\left(1 + \frac{1}{n}\right)^2 a_n = 2 \cdot \frac{(n+1)^2}{n^2} a_n$

$\therefore \frac{a_{n+1}}{(n+1)^2} = 2 \cdot \frac{a_n}{n^2} \quad \therefore \left\{ \frac{a_n}{n^2} \right\}$ 是公比为2的等比数列

$\therefore \frac{a_n}{n^2} = \left(\frac{a_1}{1^2} \right) \cdot 2^{n-1} = 2^n$

$\therefore a_n = n^2 \cdot 2^n$

(2) 思路: $c_n = \frac{n}{a_n} = \frac{1}{n \cdot 2^n}$, 无法直接求和, 所以考虑放缩成为可求和的通项公式 (不等

号: <), 若要放缩为裂项相消的形式, 那么需要构造出“顺序同构”的特点。观察分母中有 n , 故分子分母通乘以 $(n-1)$, 再进行放缩调整为裂项相消形式。

解: $c_n = \frac{n}{a_n} = \frac{1}{n \cdot 2^n} = \frac{n-1}{n(n-1)2^n}$

而 $\frac{1}{(n-1)2^{n-1}} - \frac{1}{n \cdot 2^n} = \frac{2n-(n-1)}{n(n-1)2^n} = \frac{n+1}{n(n-1)2^n}$

所以 $c_n = \frac{n-1}{n(n-1)2^n} < \frac{n+1}{n(n-1)2^n} = \frac{1}{(n-1)2^{n-1}} - \frac{1}{n \cdot 2^n} (n \geq 2)$

$$\begin{aligned} c_1 + c_2 + \cdots + c_n &< c_1 + c_2 + c_3 + \left(\frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \frac{1}{4 \cdot 2^4} - \frac{1}{5 \cdot 2^5} + \cdots + \frac{1}{(n-1)2^{n-1}} - \frac{1}{n \cdot 2^n} \right) \\ &= \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{24} - \frac{1}{n \cdot 2^n} = \frac{17}{24} - \frac{1}{n \cdot 2^n} < \frac{17}{24} \quad (n > 3) \end{aligned}$$

$\therefore c_n > 0 \quad \therefore c_1 < c_1 + c_2 < c_1 + c_2 + c_3 = \frac{16}{24} < \frac{17}{24}$

3.解: (1) 在 $S_n = na_n - 3n(n-1), n \in N^*$ 中, 令 $n=2, n=3$ 可得:

$$\begin{cases} a_1 + a_2 = 2a_2 - 6 \\ a_1 + a_2 + a_3 = 3a_3 - 18 \end{cases} \Rightarrow \begin{cases} a_2 - a_1 = 6 \\ a_1 + a_2 = 16 \end{cases}$$

$\therefore a_1 = 5, a_2 = 11$

(2) $S_n = na_n - 3n(n-1) \quad \text{①}$

$S_{n-1} = (n-1)a_{n-1} - 3(n-1)(n-2) \quad \text{②}$

① - ②可得:

$$a_n = na_n - (n-1)a_{n-1} - 6(n-1) \Rightarrow (n-1)a_n = (n-1)a_{n-1} + 6(n-1) \quad (n \geq 2)$$

$$\therefore a_n = a_{n-1} + 6$$

$\therefore \{a_n\}$ 是公差为 6 的等差数列

$$\therefore a_n = a_1 + 6(n-1) = 6n - 1$$

$$\therefore S_n = na_n - 3n(n-1) = n(6n-1) - 3n(n-1) = 3n^2 + 2n$$

$$(3) \text{ 由 (2) 可得: } b_n = \sqrt{\frac{n}{3n^2 + 2n}} = \frac{1}{\sqrt{3n+2}}$$

$$\therefore b_n = \frac{1}{\sqrt{3n+2}} = \frac{2}{2\sqrt{3n+2}} < \frac{2}{\sqrt{3n+2} + \sqrt{3n-1}} = \frac{3}{2}(\sqrt{3n+2} - \sqrt{3n-1})$$

$$\begin{aligned} \therefore T_n = b_1 + b_2 + \cdots + b_n &< \frac{2}{3}[(\sqrt{5} - \sqrt{2}) + (\sqrt{8} - \sqrt{5}) + (\sqrt{3n+2} - \sqrt{3n-1})] \\ &= \frac{2}{3}(\sqrt{3n+2} - \sqrt{2}) < \frac{2}{3}\sqrt{3n+2} \end{aligned}$$

$$4. \text{解: (1) } a_n = \frac{a_{n-1}}{(-1)^n a_{n-1} - 2} \Rightarrow \frac{1}{a_n} = \frac{(-1)^n a_{n-1} - 2}{a_{n-1}} = (-1)^n - \frac{2}{a_{n-1}}$$

$$\therefore \frac{1}{a_n} + (-1)^n = 2 \cdot (-1)^n - \frac{2}{a_{n-1}} \Rightarrow \therefore \frac{1}{a_n} + (-1)^n = (-2) \cdot \left[(-1)^{n-1} + \frac{2}{a_{n-1}} \right]$$

$\therefore \left\{ \frac{1}{a_n} + (-1)^n \right\}$ 为公比是 -2 的等比数列

(2) 思路: 首先由 (1) 可求出 $\{a_n\}$ 的通项公式 $\therefore a_n = \frac{1}{3 \cdot (-2)^{n-1} - (-1)^n}$, 对于

$$\sin \frac{(2n-1)\pi}{2} \text{ 可发现 } n \text{ 为奇数时, } \sin \frac{(2n-1)\pi}{2} = 1, \quad n \text{ 为偶数时, } \sin \frac{(2n-1)\pi}{2} = -1,$$

结合 $\{a_n\}$ 通项公式可将其写成 $\sin \frac{(2n-1)\pi}{2} = (-1)^{n-1}$, 从而求出 $c_n = \frac{1}{3 \cdot 2^{n-1} + 1}$, 无法直

接求和, 所以考虑对通项公式进行放缩, 可联想到等比数列, 进而 $c_n = \frac{1}{3 \cdot 2^{n-1} + 1} < \frac{1}{3 \cdot 2^{n-1}}$,

求和后与所证不等式右端常数比较后再进行调整 (需前两项不动) 即可。

解: $\frac{1}{a_1} + (-1)^1 = 3$, 由 (1) 可得:

$$\frac{1}{a_n} + (-1)^n = \left[\frac{1}{a_1} + (-1)^1 \right] \cdot (-2)^{n-1} = 3 \cdot (-2)^{n-1}$$

$$\therefore a_n = \frac{1}{3 \cdot (-2)^{n-1} - (-1)^n}$$

$$\text{而 } \sin \frac{(2n-1)\pi}{2} = (-1)^{n-1} \quad \therefore b_n = a_n \cdot \sin \frac{(2n-1)\pi}{2} = \frac{(-1)^{n-1}}{3 \cdot (-2)^{n-1} - (-1)^n} = \frac{1}{3 \cdot 2^{n-1} + 1}$$

$$\therefore b_n = \frac{1}{3 \cdot 2^{n-1} + 1} < \frac{1}{3 \cdot 2^{n-1}}$$

$$\text{当 } n \geq 3 \text{ 时, } T_n = b_1 + b_2 + \cdots + b_n < (b_1 + b_2) + \frac{1}{3 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots + \frac{1}{3 \cdot 2^{n-1}}$$

$$= \frac{1}{4} + \frac{1}{7} + \frac{\frac{1}{12} \left[1 - \left(\frac{1}{2} \right)^{n-2} \right]}{1 - \frac{1}{2}} < \frac{1}{4} + \frac{1}{7} + \frac{1}{6} = \frac{47}{84} < \frac{4}{7}$$

因为 $\{b_n\}$ 为正项数列 $\therefore T_1 < T_2 < T_3 < \cdots < T_n$

$$\therefore \forall n \in N^*, T_n < \frac{4}{7}$$

5. 解: (1) $\therefore a_{n+1} = 3a_n \quad \therefore \{a_n\}$ 为公比是 3 的等比数列

$$\therefore a_n = a_1 \cdot 3^{n-1} = 3^{n-1}$$

在 $\{b_n\}$ 中, 令 $n=1$, $2b_1 - b_1 = S_1 \cdot S_1 \Rightarrow b_1 = 1$

$$\therefore 2b_n - 1 = S_n$$

$$2b_{n-1} - 1 = S_{n-1} \quad \therefore 2b_n - 2b_{n-1} = b_n (n \geq 2) \Rightarrow b_n = 2b_{n-1}$$

$\therefore \{b_n\}$ 是公比为 2 的等比数列

$$\therefore b_n = b_1 \cdot 2^{n-1} = 2^{n-1}$$

$$(2) \text{ 证明: } \frac{1}{a_n - b_n} = \frac{1}{3^{n-1} - 2^{n-1}} < \frac{1}{3^{n-2}}$$

$$\frac{1}{a_2 - b_2} + \frac{1}{a_3 - b_3} + \cdots + \frac{1}{a_n - b_n}$$

$$< 1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-2}} = \frac{1 \cdot \left[1 - \left(\frac{1}{3} \right)^{n-1} \right]}{1 - \frac{1}{3}} = \frac{3}{2} \left[1 - \left(\frac{1}{3} \right)^{n-1} \right] < \frac{3}{2}$$