

一般数列求通项公式专题参考答案：

类型二：给出前 n 项和与通项关系求通项公式

2. 解析 由 $a_1 = \frac{1}{2}$, $S_n = n^2 a_n$, ①

$$\therefore S_{n-1} = (n-1)^2 a_{n-1}. \text{②}$$

①-②, 得 $a_n = S_n - S_{n-1} = n^2 a_n - (n-1)^2 a_{n-1}$,

即 $a_n = n^2 a_n - (n-1)^2 a_{n-1}$, 亦即 $\frac{a_n}{a_{n-1}} = \frac{n-1}{n+1} (n \geq 2)$.

$$\therefore \frac{a_n}{a_1} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} = \frac{n-1}{n+1} \cdot \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{2}{4} \cdot \frac{1}{3} = \frac{2}{n(n+1)}.$$

$$\therefore a_n = \frac{1}{n(n+1)}.$$

3. 解: $S_n = \frac{a_n(a_n+1)}{2}, S_{n-1} = \frac{a_{n-1}(a_{n-1}+1)}{2}$

两式相减, 可得: $S_n - S_{n-1} = \frac{a_n(a_n+1)}{2} - \frac{a_{n-1}(a_{n-1}+1)}{2} (n \in N^*, n \geq 2)$

$$\therefore a_n = \frac{a_n^2 - a_{n-1}^2 + a_n - a_{n-1}}{2} \Rightarrow a_n + a_{n-1} = a_n^2 - a_{n-1}^2$$

$$\therefore a_n + a_{n-1} = (a_n + a_{n-1})(a_n - a_{n-1})$$

$$\because a_n > 0 \quad \therefore a_n - a_{n-1} = 1$$

$\therefore \{a_n\}$ 是公差为 1 的等差数列

在 $S_n = \frac{a_n(a_n+1)}{2}$ 中, 令 $n=1$, 可得 $S_1 = \frac{a_1(a_1+1)}{2} \Rightarrow a_1 = 1$

$$\therefore a_n = a_1 + (n-1)d = n$$

4. 解: $\frac{4S_1}{a_1+2} + \frac{4S_2}{a_2+2} + \cdots + \frac{4S_n}{a_n+2} = S_n$ ①

$$\frac{4S_1}{a_1+2} + \frac{4S_2}{a_2+2} + \cdots + \frac{4S_{n-1}}{a_{n-1}+2} = S_{n-1} \quad (n \geq 2, n \in N^*) \quad \text{②}$$

①-②可得:

$$\frac{4S_n}{a_n+2} = a_n \Rightarrow 4S_n = a_n^2 + 2a_n, \quad n \geq 2$$

在已知等式中令 $n=1$ ，可得： $\frac{4S_1}{a_1+2}=S_1 \Rightarrow 4S_1=a_1(a_1+2)$ ③，满足上式

$$\therefore 4S_n = a_n^2 + 2a_n \quad ④$$

$$4S_{n-1} = a_{n-1}^2 + 2a_{n-1} \quad ⑤$$

两式相减可得： $4a_n = a_n^2 + 2a_n - a_{n-1}^2 - 2a_{n-1}$

$$\Leftrightarrow 2(a_n + a_{n-1}) = a_n^2 - a_{n-1}^2, \quad \because a_n^2 - a_{n-1}^2 = (a_n + a_{n-1})(a_n - a_{n-1})$$

$$\therefore a_n - a_{n-1} = 2$$

$\therefore \{a_n\}$ 为公差是 2 的等差数列，由③可解得： $a_1 = 2$

$$\therefore a_n = a_1 + (n-1)d = 2n$$

6.解： \because 当 $n \geq 2, n \in N^*$ 时， $a_n = S_n - S_{n-1}$

$$\therefore S_n - S_{n-1} = \frac{2S_n^2}{2S_n - 1} \Rightarrow 2S_n^2 - S_n - 2S_n S_{n-1} + S_{n-1} = 2S_n^2, \text{ 整理可得:}$$

$$S_{n-1} - S_n = 2S_n S_{n-1}$$

$$\therefore \frac{1}{S_n} - \frac{1}{S_{n-1}} = 2 \quad \therefore \left\{ \frac{1}{S_n} \right\} \text{ 为公差为 2 的等差数列}$$

$$\therefore \frac{1}{S_n} = \frac{1}{S_1} + (n-1) \cdot 2 = 2n-1 \quad \therefore S_n = \frac{1}{2n-1}$$

$$a_n = \begin{cases} \frac{1}{2n-1} - \frac{1}{2n-3}, n \geq 2 \\ 1, n=1 \end{cases}$$

$$7.\text{解: } S_n = \frac{1}{2} \left(a_n + \frac{1}{a_n} \right), \text{ 当 } n \geq 2, \text{ 有 } S_n = \frac{1}{2} \left(S_n - S_{n-1} + \frac{1}{S_n - S_{n-1}} \right)$$

$$\therefore 2S = S_n - S_{n-1} + \frac{1}{S_n - S_{n-1}} \Rightarrow S_n + S_{n-1} = \frac{1}{S_n - S_{n-1}}$$

$$\therefore S_n^2 - S_{n-1}^2 = 1 \quad \therefore \{S_n^2\} \text{ 为公差是 1 的等差数列}$$

$$\therefore S_n^2 = S_1^2 + (n-1) \quad \text{在 } S_n = \frac{1}{2} \left(a_n + \frac{1}{a_n} \right) \text{ 中,}$$

令 $n=1$ 可得: $S_1 = \frac{1}{2} \left(a_1 + \frac{1}{a_1} \right)$ 可解得 $a_1 = 1$

$$\therefore S_n^2 = n \quad \therefore S_n = \sqrt{n}$$

$$\therefore a_n = \begin{cases} S_n - S_{n-1}, n \geq 2 \\ S_1, n = 1 \end{cases} \Rightarrow a_n = \begin{cases} \sqrt{n} - \sqrt{n-1}, n \geq 2 \\ 1, n = 1 \end{cases}$$

拓展类:

3.解: $a_n = \frac{n}{n-1} a_{n-1} + 2n \times 3^{n-2}$

$$\therefore \frac{a_n}{n} = \frac{1}{n-1} a_{n-1} + 2 \times 3^{n-2} \text{ 即 } \frac{a_n}{n} - \frac{a_{n-1}}{n-1} = 2 \times 3^{n-2}$$

则有 $\frac{a_n}{n} - \frac{a_{n-1}}{n-1} = 2 \times 3^{n-2}$

$$\frac{a_{n-1}}{n-1} - \frac{a_{n-2}}{n-2} = 2 \times 3^{n-3}$$

\vdots

$$\frac{a_2}{2} - \frac{a_1}{1} = 2$$

累加可得: $\frac{a_n}{n} - a_1 = 2(1 + 3 + \dots + 3^{n-2}) = \frac{2(3^{n-1} - 1)}{3 - 1}$

即 $\frac{a_n}{n} = a_1 + 3^{n-1} - 1 = 3^{n-1}$

$$\therefore a_n = n \cdot 3^{n-1}$$

6.解: $a_n = \frac{3na_{n-1}}{2a_{n-1} + n - 1} \quad \therefore \frac{1}{a_n} = \frac{2a_{n-1} + n - 1}{3na_{n-1}} = \frac{2}{3n} + \frac{n-1}{3na_{n-1}}$

$$\therefore \frac{n}{a_n} = \frac{2}{3} + \frac{n-1}{3a_{n-1}} \quad \text{设 } b_n = \frac{n}{a_n}, \text{ 则 } b_n = \frac{1}{3}b_{n-1} + \frac{2}{3}, \quad b_1 = \frac{1}{a_1} = \frac{2}{3}$$

而 $b_n = \frac{1}{3}b_{n-1} + \frac{2}{3} \Rightarrow b_n - 1 = \frac{1}{3}(b_{n-1} - 1) \quad \therefore \{b_n - 1\}$ 为公比是 $\frac{1}{3}$ 的等比数列

$$\therefore b_n - 1 = (b_1 - 1) \cdot \left(\frac{1}{3}\right)^{n-1} \quad \therefore b_n = 1 - \left(\frac{1}{3}\right)^n \text{ 即 } \frac{n}{a_n} = 1 - \left(\frac{1}{3}\right)^n$$

$$\therefore a_n = \frac{n}{1 - \left(\frac{1}{3}\right)^n} = \frac{n \cdot 3^n}{3^n - 1}$$

$$7. \text{解: } (a_{n+1}-1)(a_n-1)=3[(a_n-1)-(a_{n+1}-1)]$$

$$\therefore \frac{(a_n-1)-(a_{n+1}-1)}{(a_n-1) \cdot (a_{n+1}-1)} = \frac{1}{3} \Rightarrow \frac{1}{a_{n+1}-1} - \frac{1}{a_n-1} = \frac{1}{3}$$

$$\therefore \left\{ \frac{1}{a_n-1} \right\} \text{ 是公差为 } \frac{1}{3} \text{ 的等差数列}$$

$$\therefore \frac{1}{a_n-1} = \frac{1}{a_1-1} + \frac{1}{3}(n-1) = \frac{1}{3}n + \frac{2}{3}$$

$$\therefore a_n-1 = \frac{3}{n+2} \Rightarrow a_n = \frac{n+5}{n+2}$$

$$8. \text{解: } a_n = \frac{(n-1)a_{n-1} + (n+1)a_{n+1}}{2n} \Rightarrow 2na_n = (n-1)a_{n-1} + (n+1)a_{n+1}$$

$$\therefore (n+1)(a_{n+1}-a_n) = (n-1)(a_n-a_{n-1})$$

$$\therefore \frac{(a_{n+1}-a_n)}{(a_n-a_{n-1})} = \frac{(n-1)}{(n+1)} \quad \text{设 } b_n = a_{n+1}-a_n, \text{ 即 } \frac{b_n}{b_{n-1}} = \frac{n-1}{n+1}$$

$$\therefore \frac{b_n}{b_{n-1}} \cdot \frac{b_{n-1}}{b_{n-2}} \cdots \frac{b_2}{b_1} = \frac{n-1}{n+1} \cdot \frac{n-2}{n} \cdots \frac{1}{3} \Rightarrow \frac{b_n}{b_1} = \frac{2}{n(n+1)}$$

$$\therefore b_n = \frac{2}{n(n+1)}b_1 \quad \because b_1 = a_2 - a_1 = 1$$

$$\therefore a_{n+1} - a_n = b_n = \frac{2}{n(n+1)} = 2 \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$\therefore (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1) = 2 \left(\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n-2} - \frac{1}{n-1} + \cdots + 1 - \frac{1}{2} \right)$$

$$= 2 \left(1 - \frac{1}{n} \right)$$

$$\text{即 } a_n - a_1 = 2 \left(1 - \frac{1}{n} \right) \quad \therefore a_n = 3 - \frac{2}{n}$$

思路二：本题还可以从递推公式中的“同构入手”，构造辅助数列，

$$a_n = \frac{(n-1)a_{n-1} + (n+1)a_{n+1}}{2n} \Rightarrow 2(na_n) = (n-1)a_{n-1} + (n+1)a_{n+1}, \text{ 此三项具备同构特}$$

点, 故设 $b_n = na_n$, 则递推公式变为: $2b_n = b_{n+1} + b_{n-1}$, 所以 $\{b_n\}$ 为等差数列, 其公差可由 b_1, b_2 计算, 从而得到 $\{b_n\}$ 通项公式以求得 a_n

$$\text{解: } \because a_n = \frac{(n-1)a_{n-1} + (n+1)a_{n+1}}{2n}$$

$$\therefore 2(na_n) = (n-1)a_{n-1} + (n+1)a_{n+1}$$

设 $b_n = na_n$, 则递推公式变为: $2b_n = b_{n+1} + b_{n-1}$

$\therefore \{b_n\}$ 为等差数列

$$b_1 = a_1 = 1, b_2 = 2a_2 = 4 \quad \therefore d = b_2 - b_1 = 3$$

$$\therefore b_n = b_1 + (n-1)d = 3n - 2, \text{ 即 } na_n = 3n - 2$$

$$\therefore a_n = 3 - \frac{2}{n}$$

$$9. \text{解: } a_{n+2} - 2a_{n+1} + a_n = 4 \Rightarrow (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = 4$$

设 $b_n = a_{n+1} - a_n$, 则 $b_{n+1} - b_n = 4$, 且 $b_1 = a_2 - a_1 = 2$

$\therefore \{b_n\}$ 为公差是 4 的等差数列

$$\therefore b_n = b_1 + (n-1) \cdot 4 = 4n - 2$$

$$\therefore a_{n+1} - a_n = 4n - 2$$

$$a_n - a_{n-1} = 4(n-1) - 2$$

\vdots

$$a_2 - a_1 = 4 \times 1 - 2$$

$$\therefore a_n - a_1 = 4[1 + 2 + \cdots + (n-1)] - 2(n-1)$$

$$= 4 \cdot \frac{n(n-1)}{2} - 2(n-1) = 2n^2 - 4n + 2$$

$$\therefore a_n = 2n^2 - 4n + 3$$

10.解：两边取对数得： $\log_2^{a_n} = 1 + 2\log_2^{a_{n-1}}$ ， $\log_2^{a_n} + 1 = 2(\log_2^{a_{n-1}} + 1)$ ，设 $b_n = \log_2^{a_n} + 1$ ，
则 $b_n = 2b_{n-1}$

$\{b_n\}$ 是以2为公比的等比数列， $b_1 = \log_2^1 + 1 = 1$ 。

$$b_n = 1 \times 2^{n-1} = 2^{n-1}, \log_2^{a_n} + 1 = 2^{n-1}, \log_2^{a_n} = 2^{n-1} - 1,$$

$$\therefore a_n = 2^{2^{n-1}-1}$$

13.思路：只从所给递推公式很难进行变形，所以考虑再构造一个递推公式并寻找关系：即

$$a_n + a_{n-1} = 2(n-1), (n \geq 2, n \in N^*), \text{两式相减可得：} a_{n+1} - a_{n-1} = 2, (n \geq 2, n \in N^*), \text{从}$$

而可得在 $\{a_n\}$ 中，奇数项和偶数项分别可构成公差为2的等差数列，所以

$$a_{2015} = a_1 + 1007d = 2014$$

14.【答案】C

$$\text{【解析】} \because F(x) = f\left(x + \frac{1}{2}\right) - 1 \text{ 是奇函数, } \therefore F\left(\frac{1}{2}\right) + F\left(-\frac{1}{2}\right) = 0, \text{ 令 } x = \frac{1}{2}, F\left(\frac{1}{2}\right) = f(1) - 1,$$

$$\text{令 } x = -\frac{1}{2}, F\left(-\frac{1}{2}\right) = f(0) - 1, \therefore f(0) + f(1) = 2, \therefore a_1 = f(0) + f(1) = 2,$$

$$\text{令 } x = \frac{1}{n} - \frac{1}{2}, \therefore F\left(\frac{1}{n} - \frac{1}{2}\right) = f\left(\frac{1}{n}\right) - 1, \text{ 令 } x = \frac{1}{2} - \frac{1}{n}, \therefore F\left(\frac{1}{2} - \frac{1}{n}\right) = f\left(\frac{n-1}{n}\right) - 1,$$

$$\therefore F\left(\frac{1}{n} - \frac{1}{2}\right) + F\left(\frac{1}{2} - \frac{1}{n}\right) = 0, \therefore f\left(\frac{1}{n}\right) + f\left(\frac{n-1}{n}\right) = 2, \text{ 同理可得 } f\left(\frac{2}{n}\right) + f\left(\frac{n-2}{n}\right) = 2,$$

$$f\left(\frac{3}{n}\right) + f\left(\frac{n-3}{n}\right) = 2, \therefore a_n = 2 + 2 \times \frac{n-1}{n} = n + 1 (n \in N^*),$$

故选C