

数列中的不等关系专题参考答案

$$1. \text{解: (1) } nS_{n+1} - (n+3)S_n = 0 \Rightarrow \frac{S_{n+1}}{S_n} = \frac{n+3}{n}$$

$$\therefore \frac{S_n}{S_{n-1}} \cdot \frac{S_{n-1}}{S_{n-2}} \cdot \frac{S_{n-2}}{S_{n-3}} \cdots \frac{S_2}{S_1} = \frac{n+2}{n-1} \cdot \frac{n+1}{n} \cdots \frac{4}{1}$$

$$\therefore \frac{S_n}{S_1} = \frac{(n+2)(n+1)n}{3 \cdot 2} = \frac{(n+2)(n+1)n}{6}$$

$$\therefore S_1 = a_1 = 1 \quad \therefore S_n = \frac{(n+2)(n+1)n}{6}$$

$$\therefore n \geq 2 \text{ 时, } a_n = S_n - S_{n-1} = \frac{n(n+1)(n+2)}{6} - \frac{(n-1)n(n+1)}{6} = \frac{n(n+1)}{2}$$

当 $n=1$ 时, $a_1=1$ 符合上式

$$\therefore a_n = \frac{n(n+1)}{2}$$

(2) 思路: 由 (1) 可得: $c_n = 2^n \left(\frac{2}{n+1} - \lambda \right)$, 由已知 $\{c_n\}$ 为单调递减数列可得 $c_{n+1} < c_n$

对 $\forall n \in N^*$ 均成立, 所以代入 $\{c_n\}$ 通项公式得到关于 n, λ 的不等式 $\lambda > \frac{4}{n+2} - \frac{2}{n+1}$, 即

只需 $\lambda > \left(\frac{4}{n+2} - \frac{2}{n+1} \right)_{\max}$, 构造函数或者数列求出 $\left(\frac{4}{n+2} - \frac{2}{n+1} \right)$ 的最大值即可

$$\text{解: } c_n = 2^n \left(\frac{n}{a_n} - \lambda \right) = 2^n \left(\frac{n}{\frac{n(n+1)}{2}} - \lambda \right) = 2^n \left(\frac{2}{n+1} - \lambda \right)$$

$\therefore \{c_n\}$ 是递减数列 $\therefore \forall n \in N^*, c_{n+1} < c_n$

$$\text{即 } 2^{n+1} \left(\frac{2}{n+2} - \lambda \right) < 2^n \left(\frac{2}{n+1} - \lambda \right)$$

$$\Rightarrow \frac{4}{n+2} - 2\lambda < \frac{2}{n+1} - \lambda \Rightarrow \lambda > \frac{4}{n+2} - \frac{2}{n+1}$$

$$\therefore \text{只需 } \lambda > \left(\frac{4}{n+2} - \frac{2}{n+1} \right)_{\max}$$

① 构造函数：设 $f(x) = \frac{4}{x+2} - \frac{2}{x+1} (x \geq 1)$

$$\begin{aligned} \text{则 } f'(x) &= -\frac{4}{(x+2)^2} + \frac{2}{(x+1)^2} = \frac{2(x+2)^2 - 4(x+1)^2}{(x+2)^2(x+1)^2} = \frac{4-2x^2}{(x+2)^2(x+1)^2} \\ &= -\frac{2(x-\sqrt{2})(x+\sqrt{2})}{(x+2)^2(x+1)^2} \end{aligned}$$

所以 $f(x)$ 在 $(1, \sqrt{2})$ 单调递增，在 $(\sqrt{2}, +\infty)$ 单调递减

$$f(1) = \frac{1}{3}, f(2) = \frac{1}{3} \quad \therefore n \in N^* \text{ 时, } f(n)_{\max} = f(1) = f(2) = \frac{1}{3}$$

$$\text{即 } \left(\frac{4}{n+2} - \frac{2}{n+1} \right)_{\max} = \frac{1}{3} \quad \therefore \lambda > \frac{1}{3}$$

② 构造数列：设数列 $\{t_n\}$ 的通项公式 $t_n = \frac{4}{n+2} - \frac{2}{n+1}$

$$\begin{aligned} \therefore t_n - t_{n-1} &= \frac{4}{n+2} - \frac{2}{n+1} - \left(\frac{4}{n+1} - \frac{2}{n} \right) = \frac{4}{n+2} - \frac{6}{n+1} + \frac{2}{n} \quad (n \geq 2) \\ &= \frac{4n(n+1) - 6n(n+2) + 2(n+1)(n+2)}{n(n+1)(n+2)} = \frac{4-2n}{n(n+1)(n+2)} \end{aligned}$$

$\therefore n > 2$ 时, $t_n - t_{n-1} < 0$, 即 $t_n < t_{n-1}$

当 $n=2$ 时, $t_2 = t_1$

所以 $\{t_n\}$ 的最大项为 $t_2 = t_1 = \frac{1}{3}$

$$\therefore \lambda > \frac{1}{3}$$

$$2. \text{ 解: (1) } b_3 \Rightarrow (\sqrt{2})^{b_3} = (\sqrt{2})^{6+b_2}$$

$$\therefore a_1 a_2 a_3 = a_1 a_2 \cdot (\sqrt{2})^6 \quad \therefore a_3 = 8 \quad \therefore q^2 = \frac{a_3}{a_1} = 4 \Rightarrow q = 2 \text{ 或 } q = -2 \text{ (舍)}$$

$$\therefore a_n = a_1 q^{n-1} = 2^n$$

$$\therefore (\sqrt{2})^{b_n} = a_1 \cdot a_2 \cdots a_n = 2^{1+2+\cdots+n}$$

$$\therefore 2^{\frac{b_n}{2}} = 2^{\frac{n(n+1)}{2}} \Rightarrow b_n = n(n+1)$$

$$(2) \textcircled{1} c_n = \frac{1}{a_n} - \frac{1}{b_n} = \left(\frac{1}{2}\right)^n - \frac{1}{n(n+1)} = \left(\frac{1}{2}\right)^n - \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\begin{aligned} \therefore S_n &= \left[\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \right] - \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{\frac{1}{2} \left[1 - \left(\frac{1}{2}\right)^n \right]}{1 - \frac{1}{2}} - 1 + \frac{1}{n+1} = \frac{1}{n+1} - \left(\frac{1}{2}\right)^n \end{aligned}$$

② 思路：实质是求 S_n 取到最大值的项，考虑分析 S_n 的单调性，从解析式上很难通过函数的单调性判断，从而考虑相邻项比较。对于 S_n 而言， $\{S_n\}$ 的增减受 c_n 符号的影响，所以将

问题转化为判断 c_n 的符号。 $c_n = \left(\frac{1}{2}\right)^n - \frac{1}{n(n+1)}$ 可估计出当 n 取得值较大时， c_n 会由正项

变为负项。所以只要寻找到正负的分界点即可

$$\text{解： } c_n = \left(\frac{1}{2}\right)^n - \frac{1}{n(n+1)} = \frac{1}{n(n+1)} \left(\frac{n(n+1)}{2^n} - 1 \right)$$

当 $n \leq 4$ 时，可验证 $\frac{n(n+1)}{2^n} - 1 \geq 0$ ，从而可得 $c_n \geq 0$

$$\text{设 } d_n = \frac{n(n+1)}{2^n} - 1, \text{ 则 } d_{n+1} - d_n = \frac{(n+1)(n+2)}{2^{n+1}} - \frac{n(n+1)}{2^n} = -\frac{(n+1)(n-2)}{2^{n+1}}$$

当 $n \geq 5$ 时， $d_{n+1} < d_n \Rightarrow \{d_n\}$ 递减

$$\therefore d_n \leq d_5 = \frac{5 \cdot 6}{2^5} - 1 < 0$$

$$\therefore n \geq 5 \text{ 时, } c_n < 0 \quad \therefore (S_n)_{\max} = S_4$$

$\therefore k = 4$ 时，均有 $S_4 \geq S_n$

$$3. \text{解: (1)} \quad 2nS_{n+1} - 2(n+1)S_n = n(n+1) \Rightarrow \frac{S_{n+1}}{n+1} - \frac{S_n}{n} = \frac{1}{2}$$

$\therefore \left\{ \frac{S_n}{n} \right\}$ 为公差是 $\frac{1}{2}$ 的等差数列

$$\therefore \frac{S_n}{n} = \frac{S_1}{1} + \frac{1}{2}(n-1) = \frac{n+1}{2}$$

$$\therefore S_n = \frac{n(n+1)}{2} \quad \therefore n \geq 2 \text{ 时, } a_n = S_n - S_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = n$$

$$\because a_1 = 1 \text{ 符合上式} \quad \therefore a_n = n$$

$$b_{n+2} - 2b_{n+1} + b_n = 0 \Rightarrow b_{n+2} + b_n = 2b_{n+1} \quad \therefore \{b_n\} \text{ 为等差数列}$$

$$\text{设 } \{b_n\} \text{ 前 } n \text{ 项和为 } P_n \quad \therefore P_9 = 9b_5 = 63 \quad \therefore b_5 = 7 \quad \because b_3 = 5$$

$$\therefore d = \frac{b_5 - b_3}{5 - 3} = 1$$

$$\therefore b_n = n + 2$$

$$(2) \text{ 思路: 依题意可得: } c_n = \frac{b_n}{a_n} + \frac{a_n}{b_n} = \frac{n+2}{n} + \frac{n}{n+2} = 2 + 2\left(\frac{1}{n} - \frac{1}{n+2}\right), \text{ 可求出}$$

$$T_n = 2n + 3 - 2\left(\frac{1}{n+1} + \frac{1}{n+2}\right), \text{ 从而 } T_n - 2n = 3 - 2\left(\frac{1}{n+1} + \frac{1}{n+2}\right), \text{ 若 } b-a \text{ 最小, 则 } a, b$$

应最接近 $T_n - 2n$ 的最大最小值 (或是临界值), 所以问题转化成为求 $3 - 2\left(\frac{1}{n+1} + \frac{1}{n+2}\right)$ 的

范围, 可分析其单调性。 $f(n) = 3 - 2\left(\frac{1}{n+1} + \frac{1}{n+2}\right)$ 单调递增。所以最小值为 $f(1) = \frac{4}{3}$,

而当 $n \rightarrow +\infty$ 时, $f(n) \rightarrow 3$, 所以 $f(n)$ 无限接近 3, 故 $T_n - 2n$ 的取值范围为 $\left[\frac{4}{3}, 3\right)$ 中

的离散点, 从而求出 $b-a$ 的最小值

$$\text{解: } c_n = \frac{n+2}{n} + \frac{n}{n+2} = 1 + \frac{2}{n} + \frac{n+2-2}{n+2} = 2 + 2\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\therefore T_n = 2n + 2\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+2}\right)$$

$$= 2n + 2 \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = 2n + 3 - 2 \left(\frac{1}{n+1} + \frac{1}{n+2} \right)$$

$$\therefore T_n - 2n = 3 - 2 \left(\frac{1}{n+1} + \frac{1}{n+2} \right)$$

设 $f(n) = 3 - 2 \left(\frac{1}{n+1} + \frac{1}{n+2} \right)$, 可知 $f(n)$ 递增

$$\therefore f(n) \geq f(1) = \frac{4}{3}, \text{ 当 } n \rightarrow \infty \text{ 时, } f(n) \rightarrow 3$$

$$\therefore f(n) \in \left[\frac{4}{3}, 3 \right) \quad \therefore \left[\frac{4}{3}, 3 \right) \subseteq [a, b]$$

$$\text{若 } b-a \text{ 最小, 则 } a = \frac{4}{3}, b = 3 \quad \therefore (b-a)_{\min} = \frac{5}{3}$$

$$4. \text{ 解: (1) } \frac{a_{n+1}}{a_n} - \frac{2a_n}{a_{n+1}} = 1 \Rightarrow a_{n+1}^2 - a_n a_{n+1} - 2a_n^2 = 0$$

$$\therefore (a_{n+1} + a_n)(a_{n+1} - 2a_n) = 0$$

$$\therefore a_{n+1} = -a_n \text{ (舍) 或 } a_{n+1} = 2a_n$$

$\therefore \{a_n\}$ 是公比为 2 的等比数列

$$S_5 + 2 = a_6 \Rightarrow \frac{a_1(2^5 - 1)}{2 - 1} + 2 = 2^5 a_1, \text{ 解得: } a_1 = 2$$

$$\therefore a_n = a_1 2^{n-1} = 2^n$$

(2) 思路: 由 (1) 可得 $b_n = 4^n$, 进而可求出 $T_n = \frac{4}{3}(4^n - 1)$, 比较大小只需两式作差,

再进行化简通分可得 $\frac{T_{n+1} + 12}{4T_n} - \frac{4n + 6}{4n - 1} = \frac{4(3n + 1 - 7 \cdot 4^{n-1})}{(4^n - 1)(4n - 1)}$ 。利用函数或构造数列判断

出 $3n + 1 - 7 \cdot 4^{n-1}$ 的符号即可

$$\text{解: } b_n = a_n^2 = 4^n \quad \therefore T_n = \frac{4(4^n - 1)}{4 - 1} = \frac{4}{3}(4^n - 1)$$

$$\therefore \frac{T_{n+1} + 12}{4T_n} = \frac{\frac{4}{3}(4^{n+1} - 1) + 12}{4 \cdot \frac{4}{3}(4^n - 1)} = \frac{4^{n+1} + 8}{4^{n+1} - 4} = 1 + \frac{3}{4^n - 1}$$

$$\frac{4n+6}{4n-1} = 1 + \frac{7}{4n-1}$$

$$\therefore \frac{T_{n+1}+12}{4T_n} - \frac{4n+6}{4n-1} = \left(1 + \frac{3}{4^n-1}\right) - \left(1 + \frac{7}{4n-1}\right) = \frac{3}{4^n-1} - \frac{7}{4n-1} = \frac{4(3n+1-7 \cdot 4^{n-1})}{(4^n-1)(4n-1)}$$

$$\text{设 } f(x) = 3x+1-7 \cdot 4^{x-1} (x \geq 1) \quad \therefore f'(x) = -7 \cdot 4^{x-1} \ln 4 + 3, \text{ 可得 } f'(x) < 0$$

$$\therefore f(x) \text{ 为减函数 } \therefore f(x) \leq f(1) = -3 < 0$$

$$\therefore 3n+1-7 \cdot 4^{n-1} < 0$$

$$\therefore \frac{T_{n+1}+12}{4T_n} < \frac{4n+6}{4n-1}$$

5. 思路：若 $S_{2n+1} - S_n \leq \frac{m}{10}$ 恒成立， $(S_{2n+1} - S_n)_{\max} \leq \frac{m}{10}$ ，要找 S_n ，则需先确定 a_n 的通项

公式得到 $\frac{1}{a_n}$ ： $d = \frac{a_5 - a_3}{5-3} = 4$ ，所以 $a_n = a_3 + (n-4)d = 4n-3$ ，发现 $\frac{1}{a_n} = \frac{1}{4n-3}$ 无

法直接求和， $S_{2n+1} - S_n$ 很难变为简单的表达式，所以考虑将 $\{S_{2n+1} - S_n\}$ 视为一个数列，

通过相邻项比较寻找其单调性：

$$(S_{2n+3} - S_{n+1}) - (S_{2n+1} - S_n) = (S_{2n+3} - S_{2n+1}) - (S_{n+1} - S_n)$$

$$= \frac{1}{a_{2n+3}} + \frac{1}{a_{2n+2}} - \frac{1}{a_n} = \frac{1}{8n+9} + \frac{1}{8n+5} - \frac{1}{4n-3} = \frac{-104n-87}{(8n+9)(8n+5)(4n-3)} < 0, \text{ 进而}$$

$$\{S_{2n+1} - S_n\} \text{ 单调递减}, (S_{2n+1} - S_n)_{\max} = S_3 - S_1 = a_3 + a_2 = \frac{14}{45}, \text{ 所以}$$

$$\frac{m}{10} \geq \frac{14}{45} \Rightarrow m \geq \frac{28}{9}, \text{ 从而 } m = 4$$

6. 解析：(1) $na_{n+1} = 2S_n (n \in N^*)$

$$\therefore (n-1)a_n = 2S_{n-1} (n \geq 2)$$

$$\therefore na_{n+1} - (n-1)a_n = 2a_n \Rightarrow na_{n+1} = (n+1)a_n$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{n+1}{n} (n \geq 2)$$

$$\therefore \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_3}{a_2} = \frac{n}{n-1} \cdots \frac{3}{2} \text{ 可得: } \frac{a_n}{a_2} = \frac{n}{2}$$

$$a_2 = 2S_1 = 2a_1 = 2$$

$$\therefore a_n = \frac{n}{2}a_2 = n, \text{ 验证 } n=1 \text{ 时, } a_1=1 \text{ 符合上式}$$

$$\therefore a_n = n$$

由 $b_{n+1}^2 = b_n b_{n+2}$ 可知 $\{b_n\}$ 为等比数列

$$\therefore q = \frac{b_2}{b_1} = \frac{1}{2} \quad \therefore b_n = b_1 \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$$

$$(2) \quad T_n = 1 \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{2}\right)^2 + \cdots + n \cdot \left(\frac{1}{2}\right)^n$$

$$\frac{1}{2}T_n = 1 \cdot \left(\frac{1}{2}\right)^2 + \cdots + (n-1) \cdot \left(\frac{1}{2}\right)^n + n \cdot \left(\frac{1}{2}\right)^{n+1}$$

$$\therefore \frac{1}{2}T_n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n - n \cdot \left(\frac{1}{2}\right)^{n+1} = \frac{\frac{1}{2} \left[1 - \left(\frac{1}{2}\right)^n \right]}{1 - \frac{1}{2}} - n \cdot \left(\frac{1}{2}\right)^{n+1}$$

$$\therefore T_n = 2 - \frac{n+2}{2^n}$$

故恒成立不等式为: $\lambda n T_n + 2b_n S_n > 2(\lambda n + 3b_n)$

$$\lambda n \cdot \left(2 - \frac{n+2}{2^n}\right) + 2 \cdot \left(\frac{1}{2}\right)^n \cdot \frac{n(n+1)}{2} > 2 \left[\lambda n + 3 \left(\frac{1}{2}\right)^n \right]$$

化简可得: $\lambda < \frac{n^2 + n - 6}{n^2 + 2n}$ 。所以只需 $\lambda < \left(\frac{n^2 + n - 6}{n^2 + 2n} \right)_{\min}$

$$\text{设 } f(n) = \frac{n^2 + n - 6}{n^2 + 2n} = 1 - \frac{1}{\frac{n^2 + 2n}{n+6}} = 1 - \frac{1}{n+6 + \frac{24}{n+6} - 10}$$

$$\therefore f(n)_{\min} = f(1) = -\frac{4}{3}$$

$$\therefore \lambda \in \left(-\infty, -\frac{4}{3} \right)$$

$$7、\text{解析: } (1) \because S_n = n^2 + \frac{1}{2}a_n \quad \therefore S_{n+1} = (n+1)^2 + \frac{1}{2}a_{n+1}$$

$$\therefore S_{n+1} - S_n = (n+1)^2 + \frac{1}{2}a_{n+1} - \left(n^2 + \frac{1}{2}a_n\right)$$

$$\text{即 } a_{n+1} = 2n+1 + \frac{1}{2}a_{n+1} - \frac{1}{2}a_n$$

$$\therefore a_{n+1} + a_n = 4n+2$$

$$(2) \text{ 由 (1) 可知 } a_{n+1} + a_n = 4n+2$$

$$\therefore a_{n+2} + a_{n+1} = 4(n+1) + 2, \text{ 两式相减可得: } a_{n+2} - a_n = 4$$

$\therefore \{a_n\}$ 中奇数项, 偶数项分别成公差是 4 的等差数列

$$S_n = n^2 + \frac{1}{2}a_n \text{ 中令 } n=1 \Rightarrow a_1 = 2$$

$$\text{令 } n=2 \text{ 可得: } S_2 = 4 + \frac{1}{2}a_2 \Rightarrow a_1 + a_2 = 4 + \frac{1}{2}a_2 \Rightarrow a_2 = 4$$

$$\therefore a_{2k-1} = a_1 + 4(k-1) = 4k-2 = 2(2k-1)$$

$$a_{2k} = a_2 + 4(k-1) = 4k = 2 \cdot 2k$$

综上所述可得: $a_n = 2n$

$$(3) \text{ 恒成立的不等式为: } \left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \cdots \left(1 - \frac{1}{a_n}\right) < \frac{2a^2-3}{2a\sqrt{2n+1}}$$

$$\sqrt{2n+1} \cdot \left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \cdots \left(1 - \frac{1}{a_n}\right) < \frac{2a^2-3}{2a}$$

$$\frac{2a^2-3}{2a} > \left[\sqrt{2n+1} \cdot \left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \cdots \left(1 - \frac{1}{a_n}\right) \right]_{\max}$$

$$\text{设 } b_n = \sqrt{2n+1} \cdot \left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \cdots \left(1 - \frac{1}{a_n}\right), \text{ 由 } a_n = 2n \text{ 可知 } b_n > 0$$

$$\begin{aligned} \therefore \frac{b_n}{b_{n-1}} &= \frac{\sqrt{2n+1} \cdot \left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \cdots \left(1 - \frac{1}{a_n}\right)}{\sqrt{2n-1} \cdot \left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \cdots \left(1 - \frac{1}{a_{n-1}}\right)} = \sqrt{\frac{2n+1}{2n-1}} \cdot \frac{2n-1}{2n} \\ &= \frac{\sqrt{4n^2-1}}{2n} = \sqrt{\frac{4n^2-1}{4n^2}} < 1 \end{aligned}$$

$\therefore \{b_n\}$ 为递减数列

$$\therefore (b_n)_{\max} = b_1 = \sqrt{3} \cdot \left(1 - \frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{2a^2 - 3}{2a} > \frac{\sqrt{3}}{2} \Rightarrow \frac{2a^2 - \sqrt{3}a - 3}{2a} > 0 \Rightarrow a(a - \sqrt{3}) \left(a + \frac{\sqrt{3}}{2}\right) > 0$$

$$\text{解得: } a \in \left(-\frac{\sqrt{3}}{2}, 0\right) \cup (\sqrt{3}, +\infty)$$

$$8. \text{解: (1) } a_n = S_n - S_{n-1} = \frac{n^2}{4} - \frac{(n-1)^2}{4} = \frac{1}{4}(2n-1) \quad (n \geq 2)$$

$$a_1 = S_1 = \frac{1}{4} \text{ 符合上式}$$

$$\therefore a_n = \frac{1}{4}(2n-1)$$

$$(2) \quad b_n - a_n = b_n - \frac{1}{4}(2n-1)$$

$$\text{考虑 } 3b_n - b_{n-1} = n \Leftrightarrow 3\left[b_n - \frac{1}{4}(2n-1)\right] - \left[b_{n-1} - \frac{1}{4}(2n-3)\right] = 0$$

$$\text{即 } 3(b_n - a_n) - (b_{n-1} - a_{n-1}) = 0 \quad \therefore b_n - a_n = \frac{1}{3}(b_{n-1} - a_{n-1})$$

\therefore 数列 $\{b_n - a_n\}$ 为等比数列

$$(3) \text{ 思路: 由 (2) 可求得 } \{b_n\} \text{ 通项公式 } b_n = \left(b_1 - \frac{1}{4}\right) \left(\frac{1}{3}\right)^{n-1} + \frac{1}{4}(2n-1), \text{ 但不知其单调}$$

性, 但可以先考虑必要条件以缩小 b_1 的取值范围. 若要 T_3 最小, 则最起码要比 T_2, T_4 小, 从而先求出 b_1 满足的必要条件 $-47 < b_1 < -11$ (也许最后结果是其子集), 在这个范围内可判定 $\{b_n\}$ 为递增数列, 从而能保证 T_3 最小

由 (2) 可得: $\left\{b_n - \frac{1}{4}(2n-1)\right\}$ 是公比为 $\frac{1}{3}$ 的等比数列

$$\therefore b_n - \frac{1}{4}(2n-1) = \left(b_1 - \frac{1}{4}\right) \left(\frac{1}{3}\right)^{n-1}$$

$$\therefore b_n = \left(b_1 - \frac{1}{4}\right) \left(\frac{1}{3}\right)^{n-1} + \frac{1}{4}(2n-1)$$

若要 T_3 最小, 则必然要 $\begin{cases} T_3 < T_2 \\ T_3 < T_4 \end{cases} \Rightarrow \begin{cases} T_3 - T_2 < 0 \\ T_4 - T_3 > 0 \end{cases}$ 即 $\begin{cases} b_3 < 0 \\ b_4 > 0 \end{cases}$

$$\therefore \begin{cases} b_3 = \left(b_1 - \frac{1}{4}\right)\left(\frac{1}{3}\right)^2 + \frac{5}{4} < 0 \\ b_4 = \left(b_1 - \frac{1}{4}\right)\left(\frac{1}{3}\right)^3 + \frac{7}{4} < 0 \end{cases} \Rightarrow \begin{cases} b_1 < -11 \\ b_1 > -47 \end{cases}$$

$$\therefore -47 < b_1 < -11$$

则 $b_n - b_{n-1} = \frac{1}{2} - 2\left(b_1 - \frac{1}{4}\right)\left(\frac{1}{3}\right)^n > 0$, 所以 $\{b_n\}$ 为递增数列

$\therefore b_1 < b_2 < b_3 < 0, b_n > b_{n-1} > \cdots > b_4 > 0$, 符合 T_3 最小的条件

所以 $-47 < b_1 < -11$

9. 解: (1) $a_{n+1} + a_n = 4n + 3$ ①

$$\therefore a_n + a_{n-1} = 4(n-1) + 3 \quad \text{②}$$

① - ② 可得: $a_{n+1} - a_{n-1} = 4$

$\therefore \{a_n\}$ 中奇数项成等差数列, 偶数项成等差数列, 公差均为 4

$$a_1 = 2 \Rightarrow a_2 = 5$$

当 $n = 2k$ 时, $a_{2k} = a_2 + (k-1) \cdot 4 = 4k + 1$

$$\therefore a_n = 2n + 1$$

当 n 为奇数时, $a_n = 4n + 3 - a_{n+1} = 4n + 3 - [2(n+1) + 1] = 2n$

$$\therefore a_n = \begin{cases} 2n+1, n \text{ 为偶数} \\ 2n, n \text{ 为奇数} \end{cases}$$

所以当 n 为偶数时

$$\begin{aligned} S_n &= (a_1 + a_3 + \cdots + a_{n-1}) + (a_2 + a_4 + \cdots + a_n) \\ &= \frac{a_1 + a_{n-1}}{2} \cdot \frac{n}{2} + \frac{a_2 + a_n}{2} \cdot \frac{n}{2} = \frac{1}{4}n[2 + 2(n-1)] + \frac{n}{4}(5 + 2n + 1) \\ &= n^2 + \frac{3}{2}n \end{aligned}$$

n 为奇数时

$$S_n = S_{n-1} + a_n = (n-1)^2 + \frac{3}{2}(n-1) + 2n = n^2 + \frac{3}{2}n - \frac{1}{2}$$

(2) 思路：考虑将不等式转化为 a_1 的不等式，由 (1) 可得 $\{a_n\}$ 的奇数项，偶数项各为等差数列，所以只要通过分类讨论确定 n 的奇偶，即可把 a_n, a_{n+1} 均用 a_1 表示，再求出 a_1 范围即可

解：由 (1) 可得： $\{a_n\}$ 的奇数项，偶数项各为等差数列，且公差为 4

$$\text{当 } n \text{ 为奇数时, } a_n = a_1 + \left(\frac{n-1}{2}\right) \cdot 4 = a_1 + 2n - 2$$

$$a_{n+1} = 4n + 3 - a_n = 4n + 3 - (a_1 + 2n - 2) = 2n + 5 - a_1$$

$$\therefore \frac{a_n^2 + a_{n+1}^2}{a_n + a_{n+1}} \geq 4 \Rightarrow \frac{(a_1 + 2n - 2)^2 + (2n + 5 - a_1)^2}{(a_1 + 2n - 2) + (2n + 5 - a_1)} \geq 4$$

$$(a_1 + 2n - 2)^2 + (2n + 5 - a_1)^2 \geq 4(4n + 3)$$

$$\therefore a_1^2 + 2(2n - 2)a_1 + (2n - 2)^2 + a_1^2 - 2(2n + 5)a_1 + (2n + 5)^2 \geq 4(4n + 3)$$

$$\text{化简后可得: } 2a_1^2 - 14a_1 \geq -8n^2 + 4n - 17$$

$$\text{所以只需 } 2a_1^2 - 14a_1 \geq (-8n^2 + 4n - 17)_{\max}$$

$$\text{设 } f(n) = -8n^2 + 4n - 17 = -8\left(n - \frac{1}{4}\right)^2 - \frac{33}{2}$$

$$\therefore f(n)_{\max} = f(1) = -21 \quad \therefore 2a_1^2 - 14a_1 \geq -21 \text{ 解得: } a_1 \geq \frac{7+\sqrt{7}}{2} \text{ 或 } a_1 \leq \frac{7-\sqrt{7}}{2}$$

$$\text{当 } n \text{ 为偶数时, 同理: } a_{n+1} = a_1 + \frac{n}{2} \cdot 4 = a_1 + 2n, \quad a_n = 4n + 3 - a_{n+1} = 2n + 3 - a_1$$

$$\therefore \frac{a_n^2 + a_{n+1}^2}{a_n + a_{n+1}} \geq 4 \Rightarrow \frac{(-a_1 + 2n + 3)^2 + (2n + a_1)^2}{4n + 3} \geq 4$$

$$\text{化简可得: } 2a_1^2 - 6a_1 \geq -8n^2 + 4n + 3 \text{ 即 } 2a_1^2 - 6a_1 \geq (-8n^2 + 4n + 3)_{\max}$$

$$\text{设 } g(x) = -8x^2 + 4x + 3 \text{ 可得: } g(x)_{\max} = g\left(\frac{1}{4}\right) = -21$$

$$2a_1^2 - 6a_1 \geq -21 \Rightarrow 2a_1^2 - 6a_1 + 21 \geq 0 \Rightarrow a_1 \in R$$

综上所述: $a_1 \geq \frac{7+\sqrt{7}}{2}$ 或 $a_1 \leq \frac{7-\sqrt{7}}{2}$

$$\begin{aligned} 10、解: (1) a_{2k+2} &= \left(1+2\left|\cos\frac{2k\pi}{2}\right|\right)a_{2k} + \left|\sin\frac{2k\pi}{2}\right| \\ &= 3a_{2k} \end{aligned}$$

$\therefore \{a_{2k}\}$ 是公比为3的等比数列

$$(2) \text{ 当 } n=2k \text{ 时, } a_{2k} = a_2 \cdot 3^{k-1} = 3^k, \text{ 即 } a_n = 3^{\frac{n}{2}}$$

$$\begin{aligned} \text{当 } n=2k-1 \text{ 时, } a_{2k+1} &= \left(1+2\left|\cos\frac{(2k-1)\pi}{2}\right|\right)a_{2k-1} + \left|\sin\frac{(2k-1)\pi}{2}\right| \\ &= a_{2k-1} + 1 \end{aligned}$$

$\therefore \{a_{2k-1}\}$ 是公差为1的等差数列

$$\therefore a_{2k-1} = a_1 + (k-1) \cdot 1 = k \text{ 即 } a_n = \frac{n+1}{2}$$

$$\therefore a_n = \begin{cases} 3^{\frac{n}{2}}, n=2k \\ \frac{n+1}{2}, n=2k-1 \end{cases}$$

(3) 由 (2) 可得:

$$b_k = 3^k + (-1)^{k-1} \lambda \cdot 2^k \quad \therefore \text{恒成立不等式为:}$$

$$3^{k+1} + (-1)^k \lambda \cdot 2^{k+1} > 3^k + (-1)^{k-1} \lambda \cdot 2^k$$

$$\Leftrightarrow 2 \cdot 3^k > (-1)^{k-1} \lambda \cdot 3 \cdot 2^k$$

$$\Leftrightarrow (-1)^{k-1} \lambda < \left(\frac{3}{2}\right)^{k-1}$$

$$\text{当 } k \text{ 为奇数时, } \lambda < \left[\left(\frac{3}{2}\right)^{k-1}\right]_{\min} = 1$$

$$\text{当 } k \text{ 为偶数时, } \lambda > -\left[\left(\frac{3}{2}\right)^{k-1}\right]_{\max} = -\frac{3}{2}$$

$$\therefore \lambda \in \left(-\frac{3}{2}, 1\right) \quad \therefore \lambda = -1$$

