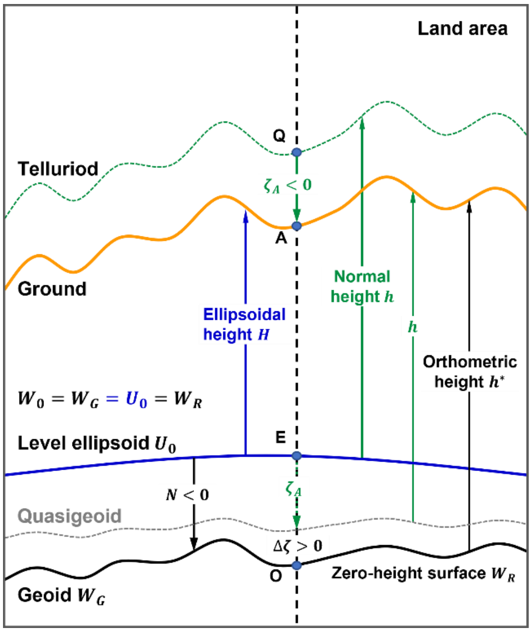
## Height system and its analytic relationship with gravity field

### 7.11.1 Relationship between the height systems and gravity field

Let the geopotential at the ground point A be $W_A$, the latitude and longitude of point Q is the same as that of point A, and its normal geopotential $U_Q$ is equal to $W_A$, then Q is located on the telluroid at A, and QA is equal to the height anomaly $\zeta_A$ at point A where the arrow downward means $\zeta_A < 0$, as shown in Figure. In the high-altitude areas, the ground height anomaly $\zeta < 0$, and the telluroid is above the ground.




Without loss of generality, let the geopotential $W_R$ of the zero-height surface in regional height datum, the geoidal geopotential $W_G$ $(= U_0)$, and the global geopotential $W_0$ (which can be understood as the geopotential of the global height datum) are exactly equal, namely:

$$W_0 = W_G = W_R \qquad (11.1)$$

In this case, the geoid is a closed surface whose geopotential is equal to the normal potential of the ellipsoidal surface, and the potential difference of the height datum is equal to zero, that is, $\delta W_R = W_0 - W_R = 0$. The difference between the geopotential $W_R$ of the zero-height surface and the geopotential $W_A$ of the ground point A is called as the geopotential number of the point A, namely $c_A = W_R - W_A = W_0 - W_A$.

The orthometric height is defined in the gravity field space, and it is the ratio of the geopotential number $c_A$ of the point A to the mean gravity $\bar{g}_A$ between the point A and the point O, and the point O is on the zero-height surface namely on the height datum surface.

$$h_A^* = \frac{c_A}{\bar{g}_A} = \frac{W_R - W_A}{\bar{g}_A} = \frac{W_0 - W_A}{\bar{g}_A} = \frac{1}{\bar{g}_A}\int_O^A g dn \qquad (11.2)$$

where $dn$ is the line element between the ground point A and the O point on the height datum surface, and $g$ is the gravity at the line element.

The normal height is defined in the normal gravity field space, which is the ratio of the normal potential number $(= U_0 - U_Q)$ of the telluroid Q at the ground point A to the mean normal gravity $\bar{\gamma}_Q$ between the Q point on the telluroid and the Q point on the normal ellipsoid surface:

$$h_A = \frac{U_0 - U_Q}{\bar{\gamma}_Q} = \frac{1}{\bar{\gamma}_Q}\int_E^Q \gamma dN \qquad (11.3)$$

where $dN$ is the line element between the point E and the Q, and $\gamma$ is the normal gravity at the line element.

The Molodensky condition assumes that the normal geopotential of the point Q located on the telluroid at point A is equal to the geopotential number of point A, that is, $U_0 - U_Q = c_A = W_0 - W_A$, and substituting it into the formula (11.3), then the normal height of Molodensky is:

$$h_A = \frac{U_0 - U_Q}{\bar{\gamma}_Q} = \frac{W_0 - W_A}{\bar{\gamma}_Q} = \frac{1}{\bar{\gamma}_Q}\int_O^A g dn \qquad (11.4)$$

Equation (11.4) is the normal height definition formula adopted by the current Chinese height system.

If the orthometric height of point A is equal to zero $h_A^* = 0$, the geopotential of point A is equal to zero $c_A = 0$ from the orthometric height definition (11.2). Substituted into the normal height definition (11.4), the normal height is also equal to zero $h_A = 0$. In this case, the geopotential of point A is equal to the geopotential of geoid $W_\mathrm{A} = W_G = W_0$, that is, point A is on the geoid. Thus it can be seen that the zero orthometric height surface, zero normal height surface and zero geopotential number surface coincide with the geoid. Therefore, whether it is the orthometric height system, normal height system, or geopotential numberl system, the height datum surface is unique, and it is the geoid.

### 7.11.2 The rigorous analytical relationship between orthometric and normal height systems

The solution of the Stokes boundary value problem is the disturbing potential of the geoid and in whole Earth space outside the geoid, that is, the Stokes boundary value problem simultaneously determines the geoidal height and the height anomaly outside the geoid, see the formular (9.1).

It is easy to find that the ground height anomaly is also the solution to the Stokes boundary value problem. In particular, the Stoke boundary value problem solution constrains the rigorous analytical relationship between the ground height anomaly $\zeta$ and the geoidal height $N$:

$$\zeta = N + \Delta\zeta = N + \int_0^{h^*} \frac{\partial \zeta}{\partial h} dh = N - \int_0^{h^*} \frac{\delta g}{\gamma} dh \qquad (11.5)$$

where $dh$ is the line element between the ground and the geoid, and $\delta g$ and $\gamma$ are the gravity disturbance (analytic gravity disturbance) and normal gravity at the line element $dh$, respectively.

$\Delta\zeta$ in formula (11.5) is the difference between the height anomaly and the geoidal height, that is, the difference between the orthometric height and the normal height. PAGravf4.5 program 5.1 can be eployed for this calculation.

According to the basic conditions for the solution of the Stokes boundary value problem, the gravity between the ground and the geoid should be equivalent to the gravity analytically continued to this point from the gravity outside the ground, which is also called as analytical gravity $g^*$. Rather than the actual gravity $g$ being affected by the terrain and surrounded by the mass, the analytical gravity $g^*$ has a strict analytical relationship with the solution of the Stokes boundary value problem. The solution condition of the Stokes boundary value problem also requires that the actual gravity and the analytical gravity are equal everywhere outside the ground.

The solution to the Stokes boundary value problem points out that the height anomaly $\zeta_o$ on the geoid is the geoidal height $N$. So it is easy to find that the zero normal height surface and the zero orthometric height surface coincide everywhere, that is the geoid, whose geopotential are equal.

In high-altitude areas, the determination accuracy of $\Delta\zeta = \zeta - N$ in formula (11.5) can be effectively improved from the observed gravity or by using regional gravity field data to refine the analytical gravity disturbance on the integral line element.

The mean gravity and integral line element gravity in the definition of orthometric height formula (11.2) replaced with analytical gravity, the strict orthometric height definition that satisfies the solution requirements of the Stokes boundary value problem is obtained:

$$h_A^* = \frac{1}{\bar{g}_A^*} \int_O^A g^* dn \qquad (11.6)$$

The zero orthometric height surface coincides with the zero normal height surface, both of which are the geoid. Obviously, only using the analytical gravity $g^*$, we can ensure that the orthometric height, normal height, height anomaly, geoid and their interrelationships are analytically compatible in Stokes boundary value theory. PAGravf4.5 calls the rigorous orthometric height as the analytical orthometric height.

### 7.11.3 The problem of quasi-geoid as height datum surface

The geoid can be uniquely determined or continuously refined according to its geopotential value, and it can be employed as the orthometric height starting surface, which meets the requirements of the uniqueness of the geodetic datum. However, it is not theoretically rigorous to regard the quasi-geoid as the normal height starting surface.

(1) The zero normal height surface is the equipotential surface whose geopotential is equal to the geopotential at the height datum zero-point. It is the geoid, not the so-called quasi-geoid.

(2) Two points with the same latitude and longitude but different heights have different height anomalies. Therefore, if the normal height is considered to start from the quasi-geoid, there must be two different starting points in the vertical direction.

(3) In most cases, the measurement points will not be just on the specific ground elevation digital model surface employed in the quasi-geoid modelling. It is necessary to add a gradient (or gravity disturbance) correction for the height anomaly at the measurement point from the quasi-geoid model. see the section 5.1 for the calculation procedure.

PAGravf4.5 downplays the concept of quasi-geoid and does not regard so-called quasi-geoid as the starting surface of normal height. The height anomalies in PAGravf4.5 are strictly in one-to-one correspondence with their spatial locations.